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STRESS SINGULARITIES IN A DISSIMILAR MATERIALS JOINT WITH EDGE TRACTIONS UNDER MECHANICAL AND THERMAL LOADINGS

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Abstract—In a two dissimilar materials joint the stresses at the intersection of the edges and the interface are singular for elastic material behaviour. For a joint with edge tractions the stresses near the singular point are the sum of singular terms and regular terms. Earlier investigations have shown that the singular stress exponents are the same for a joint with free edge and edges with tractions. In the literatures only the singular term has been studied. The emphasis in this paper is placed on giving an explicit form of the regular stress terms as a function of the edge tractions, the material properties and the geometry of the joint. It is shown that the regular terms are important also for the stress distribution near the singular point.

1. INTRODUCTION

In many technical areas dissimilar materials have to be joined together. One example is the ceramic to metal joint to combine the wear resistance, high temperature strength and thermal or electrical resistance of the ceramic with the ductility of the metal. Due to the difference in the elastic properties and the thermal expansion coefficients of the ceramic and the metal high stresses occur at the intersection of the edges and the interface of the joint under mechanical or thermal loading. Changes in temperature cause thermal stresses due to the different thermal expansion coefficients. This is especially important if joining is done at high temperatures. In such cases thermal stresses may exceed the strength of the ceramic and cause failure.

In the sense of linear elasticity, for most material combinations stress singularity exists at the intersection of the edges and the interface of the joint (denoted as singular point). In the last 10 years there were many investigations about the analysis of the stress singularity in a joint under mechanical or thermal loading. Some of them are about the dependence of the order of the singularity in the stress field near the singular point on the wedge angles and on the material constants for the joint with free edges (see Williams, 1952; Hein and Erdogan, 1971; Dempsey and Sinclair, 1981; van Vroonhoven, 1992; Bogy and Wang, 1971; Vasilopoulos, 1988; Theocaris, 1974), for the joint with edge tractions (see Bogy, 1971) and for the joint with free-fix or with fix-fix edges (see Williams, 1952; Dempsey and Sinclair, 1981). Some of them are about the stress distribution near the singular point in a joint with free edges, i.e. without tractions on the edges (Munz and Yang, 1992; Knesl *et al.*, 1991; Blanchard and Ghoniem, 1989; Blanchard and Ghoniem, 1990; Suga *et al.*, 1989). For a joint with free edge the stresses near the singular point can be described by

$$\sigma_{ij}(r,\theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_{ij0}(\theta)$$

for mechanical and thermal loadings (coordinate system see Fig. 1). The published papers showed that in the near field of the singular point also the regular term $\sigma_{ij0}(\theta)$, i.e. the r independent stress term, makes an important contribution to the stress distribution, especially, for thermal loading.

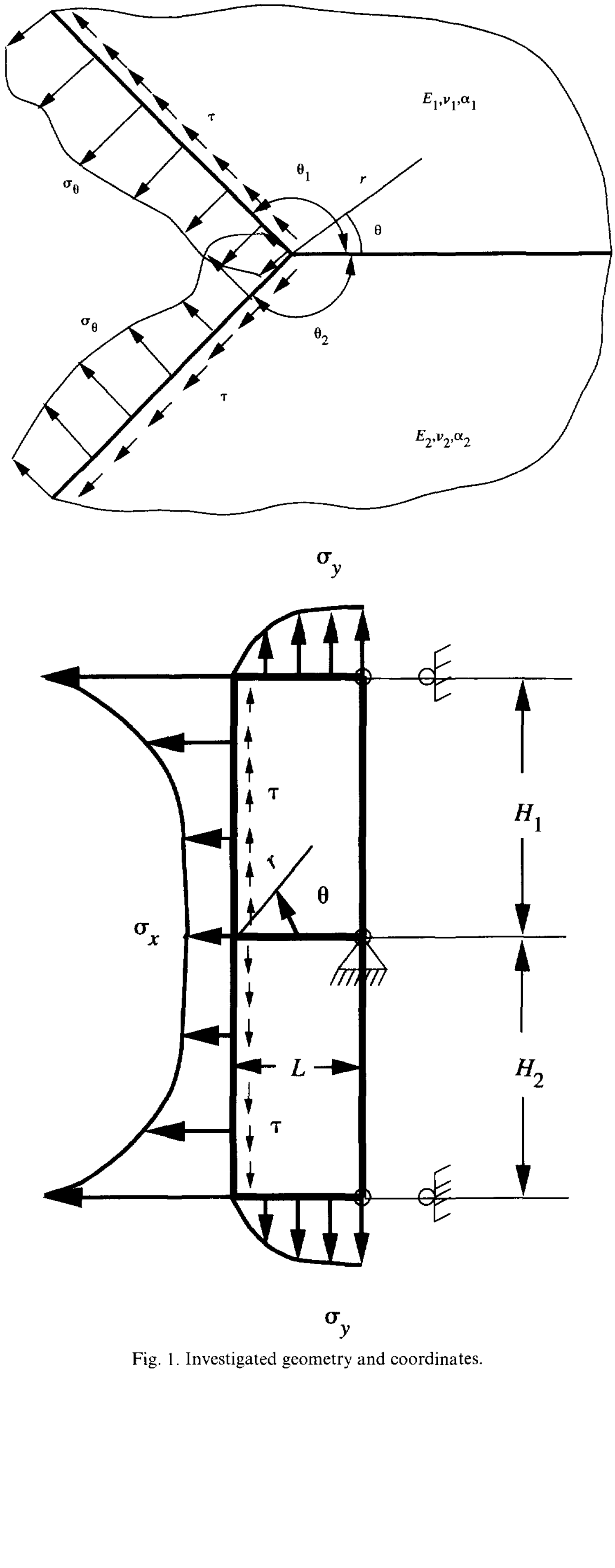

Fig. 1. Investigated geometry and coordinates.

In this paper a joint is considered (see Fig. 1a), which is subjected to three different types of loading:

(a) remote mechanical loading (R);
(b) thermal loading, e.g. a homogeneous change of temperature in the joint (TH);
(c) edge tractions (T).

For (a) and (b) the solutions are well known. The stress distribution (not only the order of the singularity) near the singular point for case (c) and for the combination of cases (a), (b) and (c) will be given in this paper. They can be described by

$$\sigma_{ij}(r,\theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_{ij0}^{\mathrm{TH}}(\theta) + \sigma_{ij0}^{\mathrm{T}}(\theta) + \sum_{l=1}^{M} (r/L)^l \sigma_{ijl}^{\mathrm{T}}(\theta).$$

All parameters in this equation, with the exception of the factor K_k, can be determined analytically. Equations for determination of the regular terms $\sigma_{ij0}^{\mathrm{T}}(\theta)$ and $\sigma_{ijl}^{\mathrm{T}}(\theta)$ will be presented in this paper. These regular terms are very important to satisfy the boundary conditions for a joint with edge tractions and they make a non-negligible contribution to the stress distribution near the singular point. The quantities ω_k, $f_{ijk}(\theta)$ and $\sigma_{ij0}^{\mathrm{TH}}(\theta)$ are the same as those in a joint with free edge (see Bogy, 1971; Munz and Yang, 1994; Munz *et al.*, 1993; Yang and Munz, 1992; Yang and Munz, 1994). The factor K_k receives a contribution from all three types of loading. Therefore, K_k can be broken down into

$$K_k = K_k^{\mathrm{R}} + K_k^{\mathrm{T}} + K_k^{\mathrm{TH}}.$$

Two examples for mechanical and thermal loadings will be shown to evidence the good agreement of the stress distributions calculated with the finite element method (FEM) and from this analytical form.

2. FUNDAMENTAL EQUATIONS

For a two-dimensional stress singularity problem the stress function $\Phi_j(r,\theta)$ written as

$$\Phi_j(r,\theta) = \sum_k r^{(2-\omega_k)}\{A_{jk}\sin(\omega_k\theta) + B_{jk}\cos(\omega_k\theta) + C_{jk}\sin[(2-\omega_k)\theta] + D_{jk}\cos[(2-\omega_k)\theta]\} \tag{1}$$

is normally used. The subscript j stands for the two materials ($j = 1, 2$).

The stresses can be calculated from eqn (1) by:

$$\sigma_r = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{2a}$$

$$\sigma_\theta = \frac{\partial^2\Phi}{\partial r^2} \tag{2b}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \tag{2c}$$

Substituting eqn (1) into eqn (2) yields

$$\sigma_{jr}(r,\theta) = \sum_k r^{-\omega_k}(1-\omega_k)\{A_{jk}(2+\omega_k)\sin(\omega_k\theta) + B_{jk}(2+\omega_k)\cos(\omega_k\theta)$$
$$- C_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta] - D_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta]\} \tag{3a}$$

$$\sigma_{j\theta}(r,\theta) = \sum_k r^{-\omega_k}(2-\omega_k)(1-\omega_k)\{A_{jk}\sin(\omega_k\theta)+B_{jk}\cos(\omega_k\theta)$$
$$+C_{jk}\sin[(2-\omega_k)\theta]+D_{jk}\cos[(2-\omega_k)\theta]\} \quad (3b)$$
$$\tau_{jr\theta}(r,\theta) = -\sum_k r^{-\omega_k}(1-\omega_k)\{A_{jk}\omega_k\cos(\omega_k\theta)-B_{jk}\omega_k\sin(\omega_k\theta)$$
$$+C_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta]-D_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta]\} \quad (j=1,2). \quad (3c)$$

From eqn (3) we can see that if $\omega = 0$ the stress term is independent of the distance r. But the stress term according to $\omega = 0$ cannot be obtained directly from eqn (3). They have the following form (see Munz *et al.* 1993):

$$\sigma_{jr0}(\theta) = 2(A_{j0}\theta+B_{j0}-C_{j0}\sin(2\theta)-D_{j0}\cos(2\theta)) \quad (4a)$$
$$\sigma_{j\theta0}(\theta) = 2(A_{j0}\theta+B_{j0}+C_{j0}\sin(2\theta)+D_{j0}\cos(2\theta)) \quad (4b)$$
$$\tau_{jr\theta0}(\theta) = -2(\tfrac{1}{2}A_{j0}+C_{j0}\cos(2\theta)-D_{j0}\sin(2\theta)) \quad (j=1,2). \quad (4c)$$

Therefore, for all cases stresses can be calculated from the summation of eqns (3) and (4), i.e.

$$\sigma_{jr}(r,\theta) = \sum_k r^{-\omega_k}(1-\omega_k)\{A_{jk}(2+\omega_k)\sin(\omega_k\theta)+B_{jk}(2+\omega_k)\cos(\omega_k\theta)$$
$$-C_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta]-D_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta]\}+\sigma_{jr0}(\theta) \quad (5a)$$
$$\sigma_{j\theta}(r,\theta) = \sum_k r^{-\omega_k}(2-\omega_k)(1-\omega_k)\{A_{jk}\sin(\omega_k\theta)+B_{jk}\cos(\omega_k\theta)$$
$$+C_{jk}\sin[(2-\omega_k)\theta]+D_{jk}\cos[(2-\omega_k)\theta]\}+\sigma_{j\theta0}(\theta) \quad (5b)$$
$$\tau_{jr\theta}(r,\theta) = -\sum_k r^{-\omega_k}(1-\omega_k)\{A_{jk}\omega_k\cos(\omega_k\theta)-B_{jk}\omega_k\sin(\omega_k\theta)$$
$$+C_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta]-D_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta]\}+\sigma_{jr\theta0}(\theta) \quad (5c)$$

where $\omega_k \neq 0$. Using the relations between stresses, strains and displacements, equations for determination of displacements can be obtained. For plane stress under thermal and mechanical loadings they are

$$\begin{aligned} u_j(r,\theta) = \sum_k \frac{r^{(1-\omega_k)}}{E_j}&\{A_{jk}[2(1-\nu_j)+\omega_k(1+\nu_j)]\sin(\omega_k\theta) \\ &+B_{jk}[2(1-\nu_j)+\omega_k(1+\nu_j)]\cos(\omega_k\theta) \\ &-C_{jk}(1+\nu_j)(2-\omega_k)\sin[(2-\omega_k)\theta] \\ &-D_{jk}(1+\nu_j)(2-\omega_k)\cos[(2-\omega_k)\theta]\} \\ &+r\cdot\alpha_j\cdot T+u_{j0}(r,\theta) \end{aligned} \quad (6a)$$

$$\begin{aligned} v_j(r,\theta) = \sum_k \frac{r^{(1-\omega_k)}}{E_j}&\{A_{jk}[2(1-\nu_j)+(2-\omega_k)(1+\nu_j)]\cos(\omega_k\theta) \\ &-B_{jk}[2(1-\nu_j)+(2-\omega_k)(1+\nu_j)]\sin(\omega_k\theta) \\ &-C_{jk}(1+\nu_j)(2-\omega_k)\cos[(2-\omega_k)\theta] \\ &+D_{jk}(1+\nu_j)(2-\omega_k)\sin[(2-\omega_k)\theta]\}+v_{j0}(r,\theta) \end{aligned} \quad (6b)$$

where $u_{j0}(r,\theta)$ and $v_{j0}(r,\theta)$ are displacements according to $\omega = 0$ (i.e. according to the stresses given in eqn (4)), the other terms in eqn (6) with $\omega_k \neq 0$. In eqn (6) the quantity α_j

is the thermal expansion coefficient, G is the shear modulus, E is the Young's modulus, ν is the Poisson's ratio and T is the temperature difference ($T = T_1 - T_0$). The terms $u_{j0}(r,\theta)$ and $v_{j0}(r,\theta)$ have the form (see Munz *et al.*, 1993)

$$u_{j0}(r,\theta) = \frac{2r}{E_j}\{A_{j0}(1-\nu_j)\theta + B_{j0}(1-\nu_j) - C_{j0}(1+\nu_j)\sin(2\theta) - D_{j0}(1+\nu_j)\cos(2\theta)\} \quad (6c)$$

$$v_{j0}(r,\theta) = \frac{2r}{E_j}\{-C_{j0}(1+\nu_j)\cos(2\theta) + D_{j0}(1+\nu_j)\sin(2\theta)\} + F_{j0}r - \frac{4A_{j0}}{E_j} r\ln(r) \quad (6d)$$

under the condition $u = v = 0$ for $r = 0$.

In this paper it is assumed that the tractions on the edges can be described by a polynomial. They have the following form:

$\theta = \theta_1$

$$\sigma_\theta = p_1 + \sum_{i=1}^{N_1} \bar{A}_i r^i \quad (7)$$

$$\tau_{r\theta} = t_1 + \sum_{j=1}^{N_2} \bar{B}_j r^j \quad (8)$$

$\theta = \theta_2$

$$\sigma_\theta = p_2 + \sum_{k=1}^{M_1} \bar{C}_k r^k \quad (9)$$

$$\tau_{r\theta} = t_2 + \sum_{l=1}^{M_2} \bar{D}_l r^l \quad (10)$$

that means that the tractions σ_θ, $\tau_{r\theta}$ at $\theta = \theta_1$ and $\theta = \theta_2$ can have different orders of polynomial. We take $M = \max\{N_1, N_2, M_1, M_2\}$, then the boundary conditions for this problem are:

for $\theta = \theta_1$:

$$\sigma_{1\theta}(r,\theta_1) = p_1 + \sum_{l=1}^{M} \bar{A}_l r^l \quad (11a)$$

$$\tau_{1r\theta}(r,\theta_1) = t_1 + \sum_{l=1}^{M} \bar{B}_l r^l \quad (11b)$$

for $\theta = \theta_2$:

$$\sigma_{2\theta}(r,\theta_2) = p_2 + \sum_{l=1}^{M} \bar{C}_l r^l \quad (11c)$$

$$\tau_{2r\theta}(r,\theta_2) = t_2 + \sum_{l=1}^{M} \bar{D}_l r^l \quad (11d)$$

for $\theta = 0^\circ$:

$$\sigma_{1\theta}(r,0) = \sigma_{2\theta}(r,0), \quad \tau_{1r\theta}(r,0) = \tau_{2r\theta}(r,0) \quad (11e,f)$$

$$u_1(r,0) = u_2(r,0), \quad v_1(r,0) = v_2(r,0). \quad (11g,h)$$

where $\bar{A}_l = 0$ (for $l = N_1+1$ to M), $\bar{B}_l = 0$ (for $l = N_2+1$ to M), $\bar{C}_l = 0$ (for $l = M_1+1$ to M), $\bar{D}_l = 0$ (for $l = M_2+1$ to M).

These eight conditions lead to the following eight equations for plane stress:

$$\sum_k r^{-\omega_k}(2-\omega_k)(1-\omega_k)\{A_{1k}\sin(\omega_k\theta_1)+B_{1k}\cos(\omega_k\theta_1)$$

$$+C_{1k}\sin[(2-\omega_k)\theta_1]+D_{1k}\cos[(2-\omega_k)\theta_1]\}+\sigma_{1\theta 0} = p_1+\sum_{l=1}^{M}\bar{A}_l r^l \quad (12a)$$

$$\sum_k r^{-\omega_k}(\omega_k-1)\{A_{1k}\omega_k\cos(\omega_k\theta_1)-B_{1k}\omega_k\sin(\omega_k\theta_1)$$

$$+C_{1k}(2-\omega_k)\cos[(2-\omega_k)\theta_1]-D_{1k}(2-\omega_k)\sin[(2-\omega_k)\theta_1]\}+\tau_{1r\theta 0} = t_1+\sum_{l=1}^{M}\bar{B}_l r^l \quad (12b)$$

$$\sum_k r^{-\omega_k}(2-\omega_k)(1-\omega_k)\{A_{2k}\sin(\omega_k\theta_2)+B_{2k}\cos(\omega_k\theta_2)$$

$$+C_{2k}\sin[(2-\omega_k)\theta_2]+D_{2k}\cos[(2-\omega_k)\theta_2]\}+\sigma_{2\theta 0} = p_2+\sum_{l=1}^{M}\bar{C}_l r^l \quad (12c)$$

$$\sum_k r^{-\omega_k}(\omega_k-1)\{A_{2k}\omega_k\cos(\omega_k\theta_2)-B_{2k}\omega_k\sin(\omega_k\theta_2)$$

$$+C_{2k}(2-\omega_k)\cos[(2-\omega_k)\theta_2]-D_{2k}(2-\omega_k)\sin[(2-\omega_k)\theta_2]\}+\tau_{2r\theta 0} = t_2+\sum_{l=1}^{M}\bar{D}_l r^l \quad (12d)$$

$$\sum_k r^{-\omega_k}\{(2-\omega_k)(1-\omega_k)(B_{1k}+D_{1k})-(2-\omega_k)(1-\omega_k)(B_{2k}+D_{2k})\}$$

$$+\sigma_{1\theta 0}(0)-\sigma_{2\theta 0}(0) = 0 \quad (12e)$$

$$\sum_k r^{-\omega_k}(1-\omega_k)\{A_{1k}\omega_k+C_{1k}(2-\omega_k)-A_{2k}\omega_k-C_{2k}(2-\omega_k)\}+\tau_{1r\theta 0}(0)-\tau_{2r\theta 0}(0) = 0 \quad (12f)$$

$$\sum_k r^{1-\omega_k}\{B_{1k}\mu[2(1-\nu_1)+\omega_k(1+\nu_1)]-D_{1k}\mu(1+\nu_1)(2-\omega_k)$$

$$-B_{2k}[2(1-\nu_2)+\omega_k(1+\nu_2)]+D_{2k}(1+\nu_2)(2-\omega_k)\}$$

$$+u_{10}(r,0)-u_{20}(r,0) = rT\cdot E_2(\alpha_2-\alpha_1) \quad (12g)$$

$$\sum_k r^{1-\omega_k}\{A_{1k}\mu[2(1-\nu_1)+(2-\omega_k)(1+\nu_1)]-C_{1k}\mu(1+\nu_1)(2-\omega_k)$$

$$-A_{2k}[2(1-\nu_2)+(2-\omega_k)(1+\nu_2)]+C_{2k}(1+\nu_2)(2-\omega_k)\}+v_{10}(r,0)-v_{20}(r,0) = 0 \quad (12h)$$

where $\mu = E_2/E_1$ and $\omega_k \neq 0$.

To solve eqn (12) different cases should be considered

(I) r-independent stress term;
(II) r-dependent stress terms, (a) $\omega_k = -1, -2, \ldots, -l, \ldots, -M$; (b) $0 \leqslant \omega_k \leqslant 1$.

(I) The case of the r-independent stress term.

From eqn (12) we can get equations to determine the r-independent stress term as follows:

$$\sigma_{1\theta 0}(\theta_1) = p_1 \quad (13a)$$

$$\tau_{1r\theta 0}(\theta_1) = t_1 \quad (13b)$$

$$\sigma_{2\theta 0}(\theta_2) = p_2 \quad (13c)$$

$$\tau_{2r\theta 0}(\theta_2) = t_2 \quad (13d)$$

$$\sigma_{1\theta 0}(0)-\sigma_{2\theta 0}(0) = 0 \quad (13e)$$

$$\tau_{1r\theta 0}(0)-\tau_{2r\theta 0}(0) = 0 \tag{13f}$$

$$u_{10}(r,0)-u_{20}(r,0) = rT\cdot E_2(\alpha_2-\alpha_1) \tag{13g}$$

$$v_{10}(r,0)-v_{20}(r,0) = 0. \tag{13h}$$

Solving eqn (13) the coefficients A_{j0}, B_{j0}, C_{j0}, D_{j0} in eqn (4) can be determined analytically. From eqn (13) we can see that the solution is made of two parts; one is the contribution of the mechanical tractions p_1, t_1, p_2 and t_2, the other is the contribution by thermal loading with the temperature difference T. The solution according to a constant mechanical traction is denoted as $\sigma_{ij0}^{\mathrm{T}}(\theta)$ and, according to thermal loading, as $\sigma_{ij0}^{\mathrm{TH}}(\theta)$. The term $\sigma_{ij0}^{\mathrm{TH}}(\theta)$ is known from the earlier investigations (see Munz *et al.*, 1993; Yang and Munz, 1992; Yang and Munz, 1994). Here only the solution for mechanical traction will be given, i.e. in eqn (13) there is $T = 0$.

Generally, for an arbitrary geometry θ_1 and θ_2 to determine the coefficients A_{j0}, B_{j0}, C_{j0}, D_{j0} an 8×8 system of linear equations has to be solved directly. The disadvantage of solving directly an 8×8 linear equations system is that the relationship between the solution of the eqn (13), the material properties, e.g. the Dundurs' parameters, the joint angles θ_1, θ_2 and the loading is not in an explicit form. For some special geometries, e.g. $\theta_1 = -\theta_2$ or $\theta_1-\theta_2 = 180^\circ$ $(\theta_2 < 0)$, the solution of eqn (13) can be simplified. As an example in the following the solution of eqn (13) for a joint with $\theta_1 = -\theta_2 = 90^\circ$ will be given. The coefficients in eqn (4) can be calculated from $A_{j0} = A_{j0}^*/Z$, $B_{j0} = B_{j0}^*/Z$, $C_{j0} = C_{j0}^*/Z$ and $D_{j0} = D_{j0}^*/Z$ with

$$Z = 4(\mu-1)[(\mu-1)-2\beta(\mu+1)] \tag{14a}$$

$$A_{10}^* = 2(t_1-t_2)[(\mu-1)-2\beta(\mu+1)] \tag{14b}$$

$$B_{10}^* = (1-\mu)\{p_1[2\beta(\mu+1)-(2\mu-1)]+p_2\}+\frac{\pi}{2}(t_1-t_2)(2\beta\mu+2\beta-3\mu+1) \tag{14c}$$

$$C_{10}^* = [t_1(2\mu-1)-t_2][(\mu-1)-2\beta(\mu+1)] \tag{14d}$$

$$D_{10}^* = (1-\mu)\{p_2-p_1[2\beta(\mu+1)+1]\}+\frac{\pi}{2}(t_2-t_1)(\mu+1)(2\beta+1) \tag{14e}$$

$$A_{20}^* = 2\mu(t_1-t_2)[(\mu-1)-2\beta(\mu+1)] \tag{14f}$$

$$B_{20}^* = (1-\mu)\{-\mu p_1+p_2[2\beta(\mu+1)+2-\mu]\}+\frac{\pi}{2}\mu(t_2-t_1)(2\beta\mu-\mu+2\beta+3) \tag{14g}$$

$$C_{20}^* = [t_1\mu+t_2(\mu-2)][(\mu-1)-2\beta(\mu+1)] \tag{14h}$$

$$D_{20}^* = (\mu-1)\{p_1\mu+p_2[2\beta(\mu+1)-\mu]\}+\frac{\pi}{2}\mu(t_1-t_2)(\mu+1)(2\beta-1) \tag{14i}$$

where $Z \neq 0$ and β is one Dundurs parameter. The Dundurs parameters are defined as

$$\alpha = \frac{m_2-\kappa m_1}{m_2+\kappa m_1}$$

$$\beta = \frac{(m_2-2)-\kappa(m_1-2)}{m_2+\kappa m_1}$$

with

$$\kappa = \frac{G_2}{G_1}$$

$$m = \begin{cases} \dfrac{4}{(1+\nu)} & \text{for plane stress} \\ 4(1-\nu) & \text{for plane strain.} \end{cases}$$

The relation between μ and α is

$$\mu = \frac{1+\alpha}{1-\alpha}.$$

Using the coefficients calculated with eqn (14), the stress term $\sigma^T_{ij0}(\theta)$ can be obtained from eqn (4). For the case $Z = 0$ and $A^*_{j0} \neq 0$ or $B^*_{j0} \neq 0$ or $C^*_{j0} \neq 0$ or $D^*_{j0} \neq 0$ there is a $\log(r)$ singularity. This will be discussed in a separate paper.

(II) The case of $\omega_k = -1, -2, \ldots, -l, \ldots, -M$.

For each $\omega_k = -l$ eqn (12) has the following form:

$$-A_{1l}\sin(l\theta_1)+B_{1l}\cos(l\theta_1)+C_{1l}\sin[(2+l)\theta_1]+D_{1l}\cos[(2+l)\theta_1]\} = \frac{\bar{A}_l}{(2+l)(1+l)} \tag{15a}$$

$$A_{1l}l\cos(l\theta_1)+B_{1l}l\sin(l\theta_1)-C_{1l}(2+l)\cos[(2+l)\theta_1] + D_{1l}(2+l)\sin[(2+l)\theta_1]\} = \frac{\bar{B}_l}{l+1} \tag{15b}$$

$$-A_{2l}\sin(l\theta_2)+B_{2l}\cos(l\theta_2)+C_{2l}\sin[(2+l)\theta_2]+D_{2l}\cos[(2+l)\theta_2]\} = \frac{\bar{C}_l}{(2+l)(1+l)} \tag{15c}$$

$$A_{2l}l\cos(l\theta_2)+B_{2l}l\sin(l\theta_2)-C_{2l}(2+l)\cos[(2+l)\theta_2]+D_{2l}(2+l)\sin[(2+l)\theta_2] = \frac{\bar{D}_l}{1+l} \tag{15d}$$

$$(B_{1l}+D_{1l})-(B_{2l}+D_{2l}) = 0 \tag{15e}$$

$$-A_{1l}l+C_{1l}(2+l)+A_{2l}l-C_{2l}(2+l) = 0 \tag{15f}$$

$$B_{1l}\mu[2(1-\nu_1)-l(1+\nu_1)]-D_{1l}\mu(1+\nu_1)(2+l) - B_{2l}[2(1-\nu_2)-l(1+\nu_2)]+D_{2l}(1+\nu_2)(2+l) = 0 \tag{15g}$$

$$A_{1l}\mu[2(1-\nu_1)+(2+l)(1+\nu_1)]-C_{1l}\mu(1+\nu_1)(2+l) - A_{2l}[2(1-\nu_2)+(2+l)(1+\nu_2)]+C_{2l}(1+\nu_2)(2+l) = 0. \tag{15h}$$

By solving eqn (15) the coefficients A_{jl}, B_{jl}, C_{jl}, D_{jl} can be determined analytically. Generally, for an arbitrary geometry with θ_1 and θ_2 an 8×8 system of linear equations has to be solved directly. To see the relationship between the solution of eqn (15) and the Dundurs's parameters α, β we need to describe the solution in an explicit form. For an arbitrary geometry with θ_1 and θ_2 the explicit form is very long and complicated. The coefficients for a joint with $\theta_1 = -\theta_2 = 90^\circ$ will be given in the following. We take

$$A_{jl} = \frac{A^*_{jl}}{Z_l}, \quad B_{jl} = \frac{B^*_{jl}}{Z_l}, \quad C_{jl} = \frac{C^*_{jl}}{Z_l}, \quad D_{jl} = \frac{D^*_{jl}}{Z_l}.$$

$1 = 2n-1$ for odd numbers of 1 and $1 = 2n$ for even numbers of 1. They can be calculated from

$$Z_l = 256n^2(2n+1)\{4n^2\mu^2\beta^2 - 8n^2\mu^2\beta + (4n^2-1)\mu^2 + 8n^2\mu\beta^2 - (8n^2-2)\mu + 4n^2\beta^2 + 8n^2\beta + 4n^2 - 1\} \quad (16ao)$$

$$A^*_{1l} = 32(2n+1)(-1)^n\{4n^2\mu^2\beta^2\bar{A} - (4n-1)2n\mu^2\beta\bar{A} + (2n-1)2n\mu^2\bar{A} + 8n^2\mu\beta^2\bar{A} + 2n\mu\beta\bar{A} - (4n+1)(2n-1)\mu\bar{A} + 2n\mu\beta\bar{C} - (2n-1)\mu\bar{C} + 4n^2\beta^2\bar{A} + 8n^2\beta\bar{A} + (4n^2-1)\bar{A} + 2n\beta\bar{C} + (2n-1)\bar{C}\} \quad (16bo)$$

$$B^*_{1l} = 32(2n+1)(-1)^{(n-1)}\{4n^2\mu^2\beta^2\bar{B} - (4n+1)2n\mu^2\beta\bar{B} + (2n+1)2n\mu^2\bar{B} + 8n^2\mu\beta^2\bar{B} - 2n\mu\beta\bar{B} - (4n-1)(2n+1)\mu\bar{B} - 2n\mu\beta\bar{D} + (2n+1)\mu\bar{D} + 4n^2\beta^2\bar{B} + 8n^2\beta\bar{B} + (4n^2-1)\bar{B} - 2n\beta\bar{D} - (2n+1)\bar{D}\} \quad (16co)$$

$$C^*_{1l} = 32(-1)^n\{4n^2(2n-1)\mu^2\beta^2\bar{A} - 2[(4n^2-3n+2)(2n+1)-2]\mu^2\beta\bar{A} + (4n^2-1)2n\mu^2\bar{A} + 8n^2(2n-1)\mu\beta^2\bar{A} - 2n(2n+1)\mu\beta\bar{A} - (4n^2-1)(4n-1)\mu\bar{A} - 2n(2n+1)\mu\beta\bar{C} + (4n^2-1)\mu\bar{C} + (2n-1)4n^2\beta^2\bar{A} + 8n^2(2n-1)\beta\bar{A} + (2n-1)^2(2n+1)\bar{A} - 2n(2n+1)\beta\bar{C} - (4n^2-1)\bar{C}\} \quad (16do)$$

$$D^*_{1l} = 32(-1)^n\{4n^2(2n+1)\mu^2\beta^2\bar{B} - 2[(2n+1)(4n^2-3n+2)+4n^2-2]\mu^2\beta\bar{B} + (4n^2-1)2n\mu^2\bar{B} + 8n^2(2n+1)\mu\beta^2\bar{B} + 2n(2n-1)\mu\beta\bar{B} - (4n^2-1)(4n+1)\mu\bar{B} + 2n(2n-1)\mu\beta\bar{D} - (4n^2-1)\mu\bar{D} + 4n^2(2n+1)\beta^2\bar{B} + 8n^2(2n+1)\beta\bar{B} + (2n-1)(2n+1)^2\bar{B} + 2n(2n-1)\beta\bar{D} + (4n^2-1)\bar{D}\} \quad (16eo)$$

$$A^*_{2l} = 32(2n+1)(-1)^n\{2n\mu^2\beta\bar{A} - (2n-1)\mu^2\bar{A} - 4n^2\mu^2\beta^2\bar{C} + 8n^2\mu^2\beta\bar{C} - (4n^2-1)\mu^2\bar{C} + 2n\mu\beta\bar{A} + (2n-1)\mu\bar{A} - 8n^2\mu\beta^2\bar{C} + 2n\mu\beta\bar{C} + (4n+1)(2n-1)\mu\bar{C} - 4n^2\beta^2\bar{C} - 2n(4n-1)\beta\bar{C} - 2n(2n-1)\bar{C}\} \quad (16fo)$$

$$B^*_{2l} = 32(2n+1)(-1)^n\{2n\mu^2\beta\bar{B} - (2n+1)\mu^2\bar{B} + 4n^2\mu^2\beta^2\bar{D} - 8n^2\mu^2\beta\bar{D} + (4n^2-1)\mu^2\bar{D} + 2n\mu\beta\bar{B} + (2n+1)\mu\bar{B} + 8n^2\mu\beta^2\bar{D} + 2n\mu\beta\bar{D} - (4n-1)(2n+1)\mu\bar{D} + 4n^2\beta^2\bar{D} + 2n(4n+1)\beta\bar{D} + 2n(2n+1)\bar{D}\} \quad (16go)$$

$$C^*_{2l} = 32(-1)^{(n-1)}\{2n(2n+1)\mu^2\beta\bar{A} - (4n^2-1)\mu^2\bar{A} + 4n^2(2n-1)\mu^2\beta^2\bar{C} - 8n^2(2n-1)\mu^2\beta\bar{C} + (2n-1)^2(2n+1)\mu^2\bar{C} + 2n(2n+1)\mu\beta\bar{A} + (4n^2-1)\mu\bar{A} + 8n^2(2n-1)\mu\beta^2\bar{C} + 2n(2n+1)\mu\beta\bar{C} - (4n-1)(4n^2-1)\mu\bar{C} + 4n^2(2n-1)\beta^2\bar{C} + 2[(2n+1)(4n^2-3n+2)-2]\beta\bar{C} + 2n(4n^2-1)\bar{C}\} \quad (16ho)$$

$$D^*_{2l} = 32(-1)^n\{2n(2n-1)\mu^2\beta\bar{B} - (4n^2-1)\mu^2\bar{B} - 4n^2(2n+1)\mu^2\beta^2\bar{D} + 8n^2(2n+1)\mu^2\beta\bar{D} - (2n-1)(2n+1)^2\mu^2\bar{D} + 2n(2n-1)\mu\beta\bar{B} + (4n^2-1)\mu\bar{B} - 8n^2(2n+1)\mu\beta^2\bar{D} + 2n(2n-1)\mu\beta\bar{D} + (4n^2-1)(4n+1)\mu\bar{D} - 4n^2(2n+1)\beta^2\bar{D} - 2[(2n+1)(4n^2-3n+2)+4n^2-2]\beta\bar{D} - 2n(4n^2-1)\bar{D}\} \quad (16io)$$

for the odd numbers of 1 and

$$Z_l = 256(n+1)^2 2n\{4n(n+1)\mu^2\beta^2 - 2(2n+1)^2\mu^2\beta + (4n(n+1)+1)\mu^2 + (2(2n+1)^2-2)\mu\beta^2 - 2(2n+1)^2\mu + 4n(n+1)\beta^2 + 2(2n+1)^2\beta + 4n(n+1) + 1\} \quad (16ae)$$

$$A^*_{1l} = 128\frac{(n+1)^2}{(2n+1)}(-1)^n\{4n(n+1)\mu^2\beta^2\bar{A}-2n(4n+3)\mu^2\beta\bar{A}+(2n+1)2n\mu^2\bar{A}$$
$$+8n(n+1)\mu\beta^2\bar{A}+2(n+1)\mu\beta\bar{A}-(4n+1)(2n+1)\mu\bar{A}-2(n+1)\mu\beta\bar{C}$$
$$+(2n+1)\mu\bar{C}+4n(n+1)\beta^2\bar{A}+2(2n+1)^2\beta\bar{A}$$
$$+(4n(n+1)+1)\bar{A}-2(n+1)\beta\bar{C}-(2n+1)\bar{C}\} \tag{16be}$$

$$B^*_{1l} = 128\frac{(n+1)n}{(2n+1)}(-1)^n\{4n(n+1)\mu^2\beta^2\bar{B}-2[n(4n+5)+1]\mu^2\beta\bar{B}+(2n+1)2(n+1)\mu^2\bar{B}$$
$$+8n(n+1)\mu\beta^2\bar{B}-2n\mu\beta\bar{B}-(4n+3)(2n+1)\mu\bar{B}+2n\mu\beta\bar{D}$$
$$-(2n+1)\mu\bar{D}+4n(n+1)\beta^2\bar{B}+2(2n+1)^2\beta\bar{B}$$
$$+(4n(n+1)+1)\bar{B}+2n\beta\bar{D}+(2n+1)\bar{D}\} \tag{16ce}$$

$$C^*_{1l} = 128\frac{(n+1)n}{(2n+1)}(-1)^n\{4n(n+1)\mu^2\beta^2\bar{A}-2[n(4n+5)+2]\mu^2\beta\bar{A}$$
$$+2(n+1)(2n+1)\mu^2\bar{A}+8n(n+1)\mu\beta^2\bar{A}-2(n+1)\mu\beta\bar{A}-(4n+3)(2n+1)\mu\bar{A}$$
$$+2(n+1)\mu\beta\bar{C}-(2n+1)\mu\bar{C}+(n+1)4n\beta^2\bar{A}+2(4n(n+1)+1)\beta\bar{A}$$
$$+(4n(n+1)+1)\bar{A}+2(n+1)\beta\bar{C}+(2n+1)\bar{C}\} \tag{16de}$$

$$D^*_{1l} = 128\frac{(n+1)n}{(2n+1)}(-1)^{(n-1)}\{4n(n+1)\mu^2\beta^2\bar{B}-2(4n^2+3n+1)\mu^2\beta\bar{B}$$
$$+(2n+1)2n\mu^2\bar{B}+8n(n+1)\mu\beta^2\bar{B}+2n\mu\beta\bar{B}-(2n+1)(4n+1)\mu\bar{B}$$
$$-2n\mu\beta\bar{D}+(2n+1)\mu\bar{D}+4n(n+1)\beta^2\bar{B}+2(4n(n+1)+1)\beta\bar{B}$$
$$+(4n(n+1)+1)\bar{B}-2n\beta\bar{D}-(2n+1)\bar{D}\} \tag{16ee}$$

$$A^*_{2l} = 128\frac{(n+1)^2}{(2n+1)}(-1)^n\{2(n+1)\mu^2\beta\bar{A}-(2n+1)\mu^2\bar{A}+4n(n+1)\mu^2\beta^2\bar{C}$$
$$-2(4n(n+1)+1)\mu^2\beta\bar{C}+(4n(n+1)+1)\mu^2\bar{C}+2(n+1)\mu\beta\bar{A}+(2n+1)\mu\bar{A}$$
$$+8n(n+1)\mu\beta^2\bar{C}-2(n+1)\mu\beta\bar{C}-(4n+1)(2n+1)\mu\bar{C}$$
$$+4n(n+1)\beta^2\bar{C}+2n(4n+3)\beta\bar{C}+2n(2n+1)\bar{C}\} \tag{16fe}$$

$$B^*_{2l} = 128\frac{n(n+1)}{(2n+1)}(-1)^n\{-2n\mu^2\beta\bar{B}+(2n+1)\mu^2\bar{B}+4n(n+1)\mu^2\beta^2\bar{D}$$
$$-2(2n+1)^2\mu^2\beta\bar{D}+(4n(n+1)+1)\mu^2\bar{D}-2n\mu\beta\bar{B}-(2n+1)\mu\bar{B}$$
$$+8n(n+1)\mu\beta^2\bar{D}+2n\mu\beta\bar{D}-(4n+3)(2n+1)\mu\bar{D}$$
$$+4n(n+1)\beta^2\bar{D}+2(n(4n+5)+1)\beta\bar{D}+2(n+1)(2n+1)\bar{D}\} \tag{16ge}$$

$$C^*_{2l} = 128\frac{n(n+1)}{(2n+1)}(-1)^n\{-2(n+1)\mu^2\beta\bar{A}+(2n+1)\mu^2\bar{A}+4n(n+1)\mu^2\beta^2\bar{C}$$
$$-2(4n(n+1)+1)\mu^2\beta\bar{C}+(4n(n+1)+1)\mu^2\bar{C}-2(n+1)\mu\beta\bar{A}-(2n+1)\mu\bar{A}$$
$$+8n(n+1)\mu\beta^2\bar{C}+2(n+1)\mu\beta\bar{C}-(4n+3)(2n+1)\mu\bar{C}+4n(n+1)\beta^2\bar{C}$$
$$+2[n(4n+5)+2]\beta\bar{C}+2(n+1)(2n+1)\bar{C}\} \tag{16he}$$

$$D^*_{2l} = 128\frac{n(n+1)}{(2n+1)}(-1)^{(n-1)}\{2n\mu^2\beta\bar{B}-(2n+1)\mu^2\bar{B}+4n(n+1)\mu^2\beta^2\bar{D}$$

$$-2(4n(n+1)+1)\mu^2\beta\bar{D}+(4n(n+1)+1)\mu^2\bar{D}+2n\mu\beta\bar{B}+(2n+1)\mu\bar{B}$$
$$+8n(n+1)\mu\beta^2\bar{D}-2n\mu\beta\bar{D}-(2n+1)(4n+1)\mu\bar{D}+4n(n+1)\beta^2\bar{D}$$
$$+2[4n^2+3n+1]\beta\bar{D}+2n(2n+1)\bar{D}\} \tag{16ie}$$

for the even numbers of 1. In eqn (16), if $Z_l = 0$ and $A_{jl}^* \neq 0$ or $B_{jl}^* \neq 0$ or $C_{jl}^* \neq 0$ or $D_{jl}^* \neq 0$, there is a $\log(r)$ singularity, which will be discussed in a separate paper.

Then the stress term according to $\omega_k = -l$ is $r^l\tilde{\sigma}_{ijl}^T(\theta)$ with

$$\tilde{\sigma}_{jrl}^T(\theta) = (1+l)\{-A_{jl}(2-l)\sin(l\theta)+B_{jl}(2-l)\cos(l\theta)$$
$$-C_{jl}(2+l)\sin[(2+l)\theta]-D_{jl}(2+l)\cos[(2+l)\theta]\} \tag{17a}$$

$$\tilde{\sigma}_{j\theta l}^T(\theta) = (2+l)(1+l)\{-A_{jl}\sin(l\theta)+B_{jl}\cos(l\theta)$$
$$+C_{jl}\sin[(2+l)\theta]+D_{jl}\cos[(2+l)\theta]\} \tag{17b}$$

$$\tilde{\tau}_{jr\theta l}^T(\theta) = (1+l)\{A_{jl}l\cos(l\theta)+B_{jl}l\sin(l\theta)$$
$$-C_{jl}(2+l)\cos[(2+l)\theta]-D_{jl}(2+l)\sin[(2+l)\theta]\}. \tag{17c}$$

It should be noted that $\tilde{\sigma}_{ijl}^T(\theta)$ do not have the unit of stress, the total $r^1\tilde{\sigma}_{ijl}^T(\theta)$ has a unit of stress. Therefore, the quantity $\tilde{\sigma}_{ijl}^T(\theta)$ is normalized and the definition

$$\sigma_{ijl}^T(\theta) = L^l\tilde{\sigma}_{ijl}^T(\theta)$$

is used. Where L is a characteristic length of the joint (see Fig. 1b). Then the stress term according to $\omega_k = -l$ is $(r/L)^1\sigma_{ijl}^T(\theta)$ and $\sigma_{ijl}^T(\theta)$ has a unit of stress.

(III) The case of $0 \leqslant \omega_k \leqslant 1$.

For this case the stress term is denoted as $\sigma_{ijl}^s(r,\theta)$ because for this ω_k stress singularity exists. The stress term according to $0 \leqslant \omega_k \leqslant 1$ has the same form as that for the same joint with a free edge (see Yang and Munz, 1995), i.e.

$$\sigma_{ij}^s(r,\theta) = \sum_{k=1}^{N}\frac{K_k}{(r/L)^{\omega_k}}f_{ijk}(\theta) \tag{18a}$$

for the real eigenvalue of the problem and

$$\sigma_{ij}^s(r,\theta) = \sum_{k=1}^{N}\frac{K_k}{(r/L)^{\omega_k}}\{\cos[p_k\ln(r/L)]f_{ijk}^c(\theta)+\sin[p_k\ln(r/L)]f_{ijk}^s(\theta)\} \tag{18b}$$

for the complex eigenvalue of the problem. In eqn (18) ω_k is the real part of the eigenvalue, p_k is the imaginary part of the eigenvalue, f_{ijk}, f_{ijk}^c and f_{ijk}^s are angular functions, N is the number of the singular terms, L is a characteristic length of the joint (see Fig. 1b). All parameters in eqn (18), except the factor K_k (which is called the stress intensity factor), can be determined analytically. Generally, the stress intensity factor should be determined by a numerical method, e.g. the finite element method (FEM) (see Munz and Yang, 1993). In eqn (18) the factor K_k receives a contribution from all three types of loading. Therefore, K_k can be separated as

$$K_k = K_k^{\mathrm{R}}+K_k^{\mathrm{T}}+K_k^{\mathrm{TH}}.$$

Finally, in a two dissimilar materials joint under the three types loading the stresses near the singular point can be calculated from

$$\sigma_{ij}(r,\theta) = \sigma^{s}_{ij}(r,\theta)+\sigma^{\mathrm{TH}}_{ij0}(\theta)+\sigma^{\mathrm{T}}_{ij0}(\theta)+\sum_{l=1}^{M}(r/L)^{l}\sigma^{\mathrm{T}}_{ijl}(\theta)$$

$$= \sum_{k=1}^{N}\frac{K_k}{(r/L)^{\omega_k}}f_{ijk}(\theta)+\sigma^{\mathrm{TH}}_{ij0}(\theta)+\sigma^{\mathrm{T}}_{ij0}(\theta)+\sum_{l=1}^{M}(r/L)^{l}\sigma^{\mathrm{T}}_{ijl}(\theta) \tag{19}$$

for the real eigenvalue. In eqn (19) the terms $\sigma^{s}_{ij}(r,\theta)$ and $\sigma^{\mathrm{TH}}_{ij0}(\theta)$ are the same as those for the same joint with free edge but with different K-factor, the term $\sigma^{\mathrm{T}}_{ij0}(\theta)$ can be calculated from eqns (4) and (14), the term $\sigma^{\mathrm{T}}_{ijl}(r,\theta)$ can be obtained from eqns (16) and (17) for a joint with $\theta_1 = -\theta_2 = 90^{\circ}$.

3. EXAMPLES

In this section two examples will be presented for a joint with edge tractions under remote mechanical loading and thermal loading to show how eqn (19) is used to describe the stress distribution near the singular point. The geometry of the joint used and the coordinates are given in Fig. 1b, where $H_1/L = H_2/L = 2$. The material properties are:

$$E_1 = 3300\,\mathrm{MPa},\quad \nu_1 = 0.35,\quad \alpha_1 = 2.5\times10^{-6}/K$$

$$E_2 = 330\,\mathrm{MPa},\quad \nu_2 = 0.35,\quad \alpha_2 = 8.5\times10^{-6}/K.$$

The assumed edge tractions are

for $\theta = \theta_1 = 90^{\circ}$

$$\sigma_x = 1+2r+3r^2+4r^3+5r^4+6r^5\ \mathrm{MPa},$$

$$\tau = 1\ \mathrm{MPa} \tag{20a}$$

and

for $\theta = \theta_2 = -90^{\circ}$

$$\sigma_x = 1+2r+3r^2+4r^3+5r^4+6r^5\ \mathrm{MPa},$$

$$\tau = -1\ \mathrm{MPa}. \tag{20b}$$

For the given tractions the terms $\sigma^{\mathrm{T}}_{ij0}(\theta)$ and $\sigma^{\mathrm{T}}_{ijl}(\theta)$ in eqn (19) have been calculated from eqns (4), (14), (16) and (17) and are shown in Tables 1–3 for components σ_r, σ_θ and $\tau_{r\theta}$ in different directions (for $L = 1$ and plane strain).

Table 1. The terms $\sigma^{\mathrm{T}}_{r0}(\theta)$ (l = 0) and $\sigma^{\mathrm{T}}_{rl}(\theta)$ in different directions (in MPa)

θ in [°]	0	45	90	−45	−90
l = 0	2.5708	0.8194	−3.0741	−0.4028	0.7656
l = 1, in 2×	0.0	0.1833	0.4815	0.5238	−0.4815
l = 2, in 3×	−0.2593	0.2593	−0.2593	0.2593	−0.2593
l = 3, in 4×	0.0	0.2476	−0.3501	0.4595	−0.6499
l = 4, in 5×	0.1489	0.2128	−0.5745	0.2128	−0.5745
l = 5, in 6×	0.0	0.0104	−0.5941	−0.2461	−0.7392

Table 2. The terms $\sigma^{\mathrm{T}}_{\theta0}(\theta)$ (l = 0) and $\sigma^{\mathrm{T}}_{\theta l}(\theta)$ in different directions (in MPa)

θ in [°]	0	45	90	−45	−90
l = 0	0.4166	0.5972	1.0	1.8194	1.0
l = 1, in 2×	0.0	0.8642	1.0	0.1571	1.0
l = 2, in 3×	−0.4815	−0.2593	1.0	−0.2593	1.0
l = 3, in 4×	0.0	−0.7071	1.0	−0.7071	1.0
l = 4, in 5×	0.2766	−0.6383	1.0	−0.6383	1.0
l = 5, in 6×	0.0	−0.2974	1.0	0.0617	1.0

Table 3. The terms $\tau^{T}_{r\theta 0}(\theta)$ $(1 = 0)$ and $\tau^{T}_{r\theta l}(\theta)$ in different directions (in MPa)

θ in [°]	0	45	90	−45	−90
$1 = 0$	1.2222	−0.9259	1.0	0.2283	−1.0
$1 = 1$, in $2\times$	−0.4815	−0.1833	0.0	0.5238	0.0
$1 = 2$, in $3\times$	0.0	−0.3704	0.0	0.3704	0.0
$1 = 3$, in $4\times$	0.2998	−0.2120	0.0	−0.2120	0.0
$1 = 4$, in $5\times$	0.0	0.3617	0.0	−0.3617	0.0
$1 = 5$, in $6\times$	−0.2177	0.5636	0.0	−0.6149	0.0

Table 4. The angular functions $f_{ij}(\theta)$ in different directions

θ in [°]	0	45	90	−45	−90
$f_r(\theta)$	−0.2487	0.4787	2.0443	0.8244	0.9809
$f_\theta(\theta)$	1.0	0.6143	0.0	0.3826	0.0
$f_{r\theta}(\theta)$	−0.2768	0.6345	0.0	−0.4809	0.0

The stress exponent ω and the angular functions $f_{ijk}(\theta)$ in eqn (19) can be obtained from the equations given in the paper (see Munz and Yang, 1994). For this geometry $(\theta_1 = -\theta_2 = 90°)$ and material combination there is only one singular term in eqn (19) and the stress exponent (i.e. the order of singularity) is $\omega = 0.2259$. The angular functions $f_{ij}(\theta)$ in different directions are shown in Table 4.

To determine the stress intensity factor K in eqn (19), the finite element method (FEM) is used for the stress analysis in the joint. In the FE-calculation the used element is normal eight nodes element and the mesh needs not be very fine. The FE-code used is ABAQUS. In eqn (19) the terms $\sigma^{TH}_{ij0}(\theta)$, σ^{T}_{ij0} and $\sigma^{T}_{ijl}(\theta)$ can be calculated analytically and the left side is known from the FE-calculation. Therefore, for one singular term eqn (19) can be rewritten as

$$\frac{K}{(r/L)^{\omega}} f_{ij}(\theta) = \sigma^{FEM}_{ij}(r,\theta) - \sigma^{TH}_{ij0}(\theta) - \sigma^{T}_{ij0}(\theta) - \sum_{l=1}^{M} (r/L)^{l} \sigma^{T}_{ijl}(\theta) \tag{21}$$

for each point with (r, θ). From eqn (21) the stress intensity factor K can be determined by using the least squares method.

3.1. *Joint under remote mechanical loading*

The assumed remote mechanical loading on the upper and lower surfaces is

$$\sigma_y = \sigma^* \sin\left(\frac{\pi}{2L} x\right)$$

with $\sigma^* = 1$ MPa. For this remote mechanical loading and the tractions given in eqn (20) the total stress intensity factor calculated from eqn (21) is $K = -4.273$ MPa. On the other hand we have made two other calculations. One is the joint with only the assumed remote mechanical loading, the corresponding factor is $K^R = 0.390$ MPa. The other is the joint with only the edge tractions, the corresponding factor is $K^T = -4.669$ MPa. For this loading (without thermal loading) the term $\sigma^{TH}_{ij0}(\theta)$ in eqn (19) is zero and therefore, there is $K^{TH} = 0$. It can be seen that the total factor K is equal to $K^{TH} + K^T + K^R$. That means that the obtained K-factors have a good accuracy, because the factors K, K^{TH}, K^T, K^R are from independent calculations.

Now with the known factor K we can calculate the stress distribution at arbitrary position near the singular point from eqn (19). A comparison of the stress distribution near

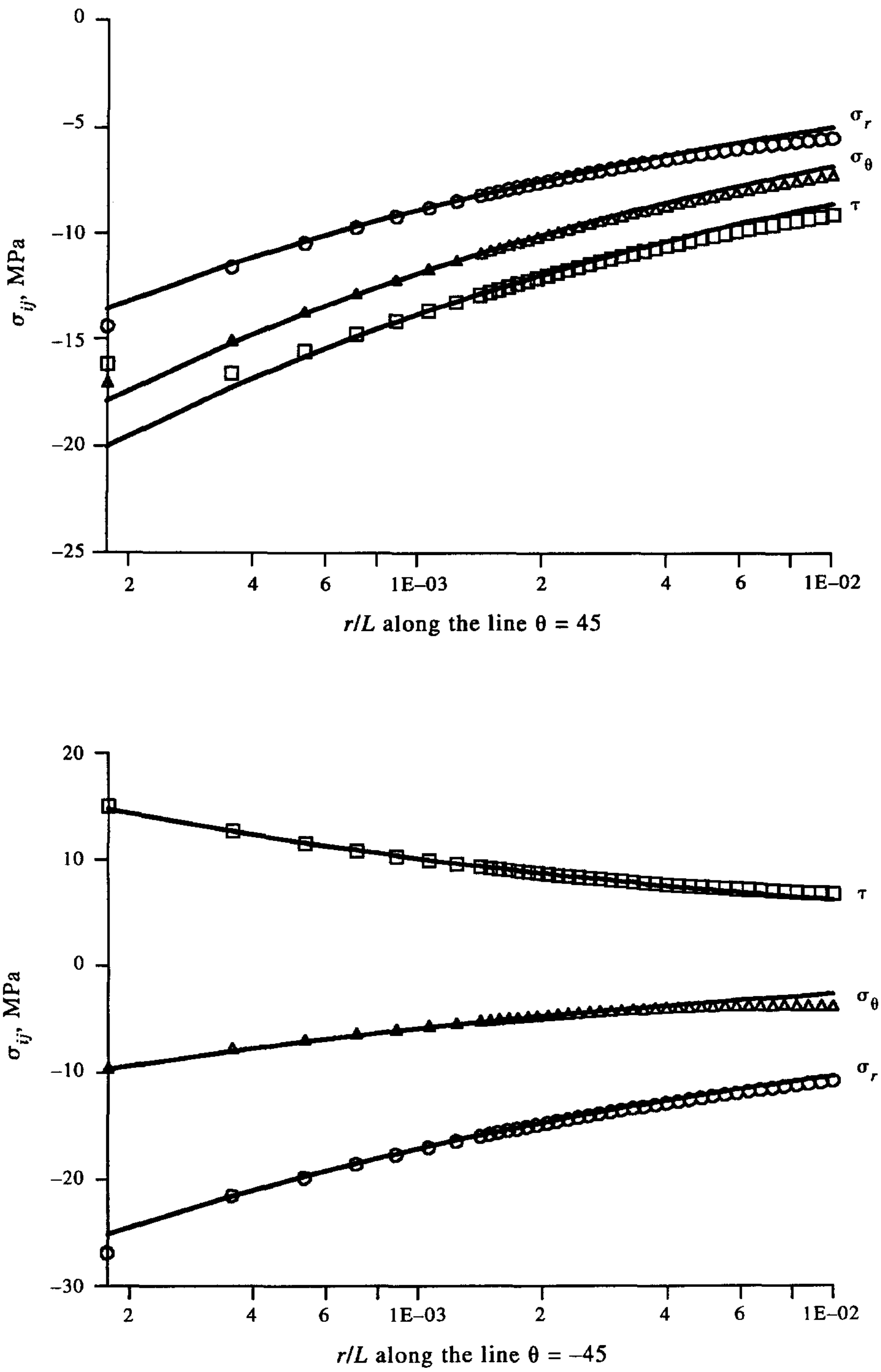

Fig. 2. A comparison of the stresses calculated from FEM (as point) and eqn (19) (as solid line) along the line with $\theta = 45^\circ$ and $\theta = -45^\circ$ for mechanical loading.

the singular point calculated from FEM and with eqn (19) is presented in Figs 2 and 3 in different directions. It can be seen that in the range of $r/L \leqslant 0.01$ they are in good agreement.

3.2. *Joint under thermal loading*

The thermal loading is a homogeneous change of temperature of $T = -200$ K in the joint. For this thermal loading the term $\sigma_{ij0}^{\mathrm{TH}}(\theta)$ in eqn (19) can be calculated from the equations given in the papers (see Munz *et al.*, 1993; Yang & Munz, 1994) and it is given

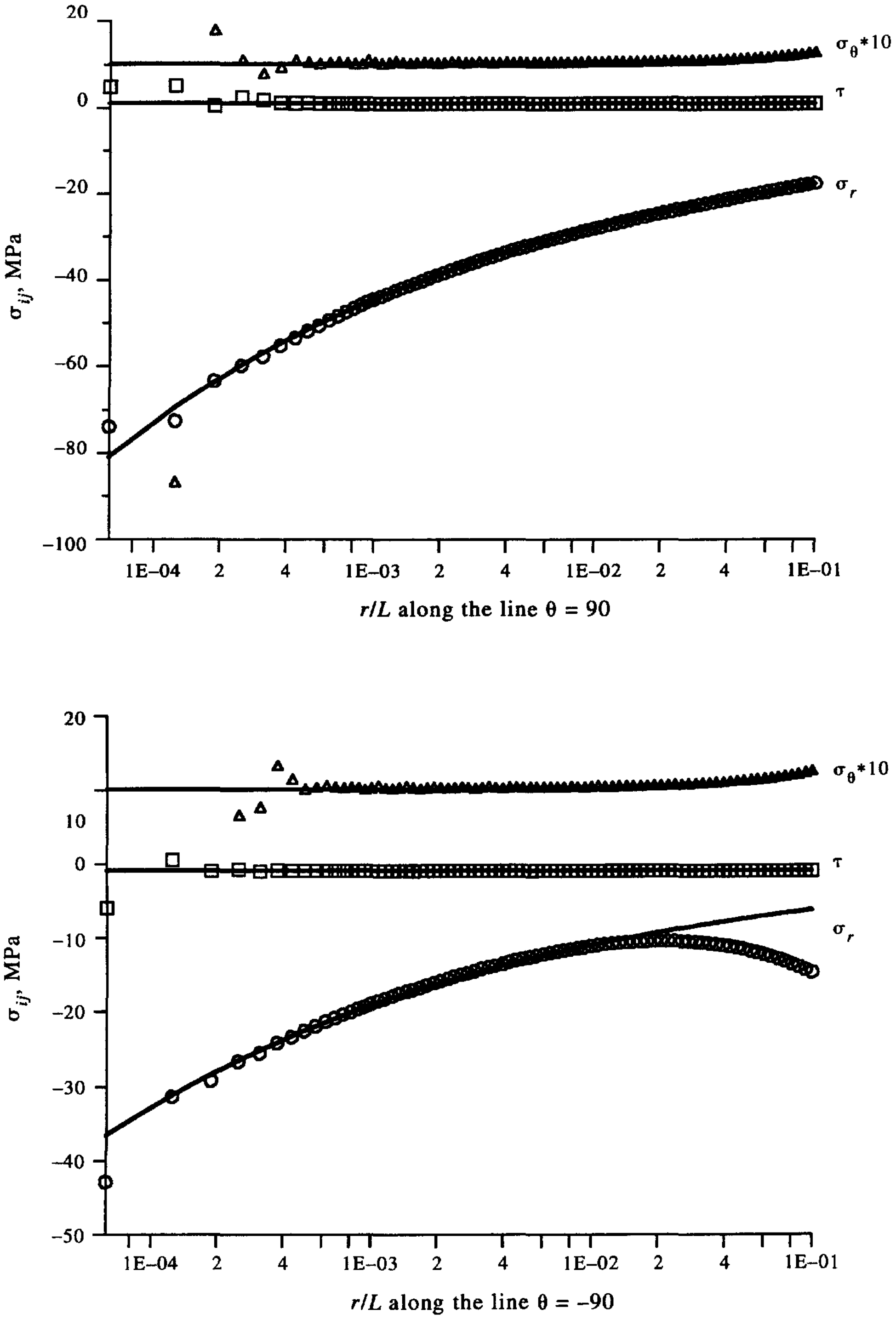

Fig. 3. A comparison of the stresses calculated from FEM (as point) and eqn (19) (as solid line) along the line with $\theta = 90°$ and $\theta = -90°$ for mechanical loading.

in Table 5. For this thermal loading and the given tractions in eqn (20) the total stress intensity factor calculated from eqn (21) is $K = -3.903$ MPa.

On the other hand we have made two other calculations. One is the joint with only temperature change (-200 K) the obtained corresponding factor is $K^{TH} = 0.759$ MPa. The other is the joint with only the edge tractions, the corresponding factor is $K^{T} = -4.669$ MPa. Here there is no remote mechanical loading therefore, $K^{R} = 0$. For this example it can be also seen that $K = K^{TH} + K^{T} + K^{R}$ is satisfied.

Table 5. The term $\sigma_{ij0}^{TH}(\theta)$ in different directions (in MPa)

θ in [°]	0	45	90	−45	−90
$\sigma_{r0}(\theta)$	0.0	−0.6286	−1.2572	−0.6286	−1.2572
$\sigma_{\theta 0}(\theta)$	−1.2572	−0.6286	0.0	−0.6286	0.0
$\sigma_{r\theta 0}(\theta)$	0.0	−0.6286	0.0	0.6286	0.0

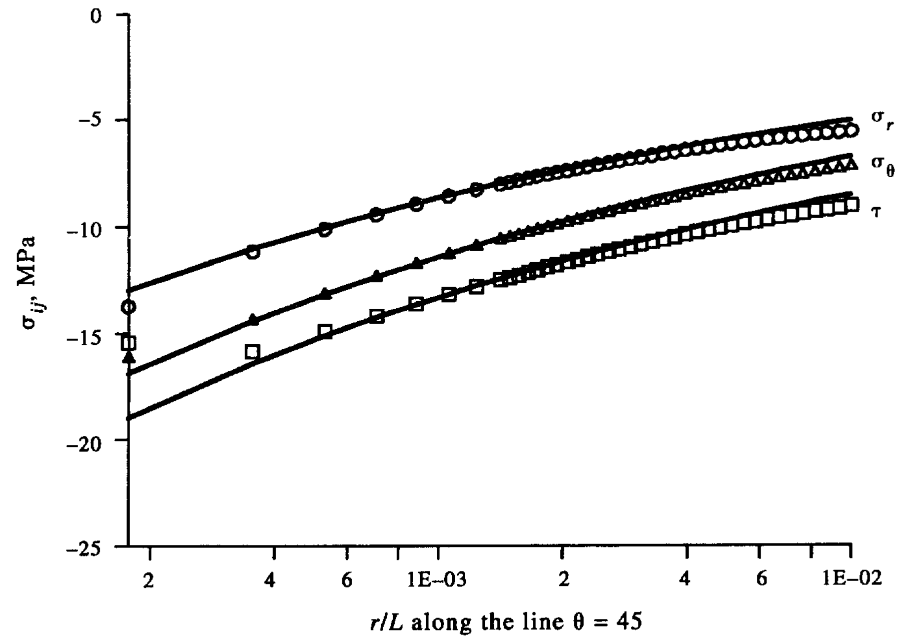

Fig. 4. A comparison of the stresses calculated from FEM (as point) and eqn (19) (as solid line) along the line with $\theta = 45°$ for thermal loading.

A comparison of the stress distribution near the singular point calculated from FEM and with eqn (19) is presented in Figs 4 and 5 in different directions. It can be seen that in the range of $r/L \leqslant 0.01$ they are in good agreement.

In eqn (19) all quantities, with the exception of K_k, can be determined analytically. For a joint subjected to three different types of loading: (a) remote mechanical loading; (b) thermal loading, e.g. a homogeneous change in temperature; (c) edge tractions, there are two ways to determine the unknown factor K.

(I) One FE-calculation for the joint with all three types loading, using eqn (21) for determination of factor K;

(II) if for each type of loading the factor K^R, K^T, K^{TH} are known we can calculate the total K-factor from $K = K^R + K^T + K^{TH}$.

In two papers (see Munz and Yang, 1992; Tilscher *et al.*, 1995) the authors have given some empirical relations to determine K^R and K^{TH} for any material combination and geometry (H_1/H_2) without requiring a FE-calculation. If K^T is known from an empirical relation the total factor K can be obtained without any further FE-calculation. Empirical relation for the factor K^T will be presented in a separate paper.

4. CONCLUSION

A joint is considered which is subjected to three different types of loading:

(a) remote mechanical loading;

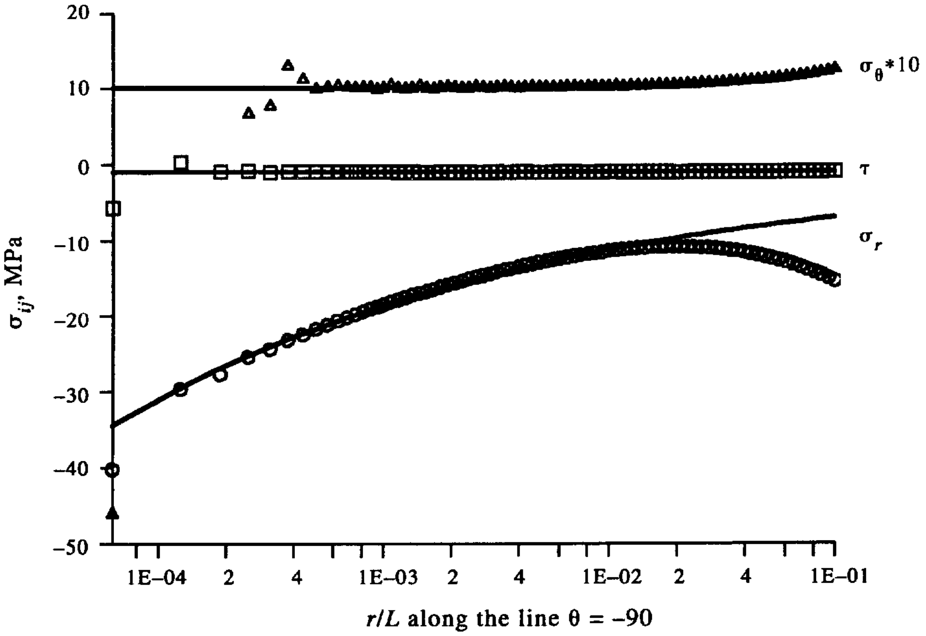

Fig. 5. A comparison of the stresses calculated from FEM (as point) and eqn (19) (as solid line) along the line with $\theta = -90°$ for thermal loading.

(b) thermal loading, e.g. a homogeneous change in temperature;
(c) edge tractions.

The stresses near the singular point in such a joint can be described by

$$\sigma_{ij}(r,\theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_{ij0}^{\mathrm{TH}}(\theta) + \sigma_{ij0}^{\mathrm{T}}(\theta) + \sum_{l=1}^{M} (r/L)^l \sigma_{ijl}^{\mathrm{T}}(\theta).$$

The quantities ω_k, $f_{ijk}(\theta)$ and $\sigma_{ij0}^{\mathrm{TH}}(\theta)$ are the same as those in a joint with a free edge and they can be determined analytically. The regular stress terms $\sigma_{ij0}^{\mathrm{T}}(\theta)$ and $\sigma_{ijl}^{\mathrm{T}}(\theta)$ are very important as well for the stress distribution near the singular point in a joint with edge tractions. Only using the singular terms to describe the stress distribution near the singular point is not enough. An explicit form for the regular stress terms, $\sigma_{ij0}^{\mathrm{T}}(\theta)$ and $\sigma_{ijl}^{\mathrm{T}}(\theta)$, is given as a function of the material properties, the geometry of the joint and the edge tractions.

Comparisons of the stress distribution near the singular point calculated from FEM and with eqn (19) have been presented. In the range of $r/L \leqslant 0.01$ they are in good agreement. The results have shown that the regular terms are very important to satisfy the boundary conditions for a joint with edge tractions and they make a non-negligible contribution to the stress distribution near the singular point.

REFERENCES

Blanchard, J. P. and Ghoniem, N. M. (1989) An eigenfunction approach to singular thermal stresses in bonded strips. *Journal of Thermal Stresses* **12,** 501–527.

Blanchard, J. P. and Ghoniem, N. M. (1990) Analysis of singular stress fields in duplex fusion components. *Journal of Nuclear Materials* **172,** 54–70.

Bogy, D. B. (1971) Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *Transactions ASME, Journal of Applied Mechanics* **38,** 377–386.

Bogy, D. B. and Wang, K. C. (1971) Stress singularities at interface corners in bonded dissimilar isotropic elastic materials. *Journal of Solids and Structures* **7,** 993–1005.

Dempsey, J. P. and Sinclair, G. B. (1981) On the singular behavior at the vertex of a bi-material wedge. *Journal of Elasticity* **11,** 317–327.

Hein, V. L. and Erdogan, F. (1971) Stress singularities in a two-material wedge. *International Journal of Fracture Mechanics* **7,** 317–330.

Knesl, Z., Sramek, A., Kad'ourek, J. and Kroupa, F. (1991) Stress concentration at the edge of coatings on tensile specimens. *Acta Technica, CSAV* **36,** 574–593.

Munz, D. and Yang, Y. Y. (1992) Stress singularities at the interface in bonded dissimilar materials under mechanical and thermal loading. *Journal of Applied Mechanics* **59,** 857–861.

Munz, D. and Yang, Y. Y. (1993) Stresses near the edge of bonded dissimilar materials described by two stress intensity factors. *International Journal of Fracture* **60,** 169–177.

Munz, D. and Yang, Y. Y. (1994) Stresses near the free edge of the interface in ceramic to metal joints. *Journal of the European Ceramic Society* **13,** 453–460.

Munz, D., Fett, T. and Yang, Y. Y. (1993) The regular stress term in bonded dissimilar materials after a change in temperature. *Engineering Fracture Mechanics* **44,** 185–194.

Suga, T., Mizuno, K. and Miyazawa, K. (1989) Thermal stresses in ceramic-to-metal joints. *MRS International I. Mtg. Advanced Materials* **8,** 137–142.

Theocaris, P. S. (1974) The order of singularity at a multi-wedge corner in a composite plate. *International Journal of Engineering Science* **12,** 107–120.

Tilscher, M., Munz, D. and Yang, Y. Y. (1995) The stress intensity factor in bonded quarter planes after a change in temperature. *Journal of Adhesion* **49,** 1–21.

Vasilopoulos, D. (1988) On the determination of higher-order terms of singular elastic stress fields near corners. *Numerical Mathematics* **53,** 51–95.

van Vroonhoven, J. C. W. (1992) Stress singularities in bi-material wedges with adhesion and delamination. *Fatigue Fracture Engineering Material Structures* **15,** 159–171.

Williams, M. L. (1952) Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *Transactions ASME, Journal of Applied Mechanics* **74,** 526–528.

Yang, Y. Y. and Munz, D. (1992) A method of determination of the regular stress term for an arbitrary joint geometry under thermal loading. KfK—Report no. 5089, Karlsruhe.

Yang, Y. Y. and Munz, D. (1994) Determination of the regular stress term in a dissimilar materials joint under thermal loading by the Mellin transformation. *Journal of Thermal Stresses* **17,** 321–336.

Yang, Y. Y. and Munz, D. (1995) The stress distribution in a dissimilar materials joint for complex singular eigenvalues under thermal loading. *Journal of Thermal Stresses* **18,** 407–419.